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COMMENTS AND REPLIES

Reply to the comment on 'Monte Carlo simulation study of the two-stage percolation transition in enhanced binary trees'

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Abstract
We discuss the nature of the two-stage percolation transition on the enhanced binary tree in order to explain the disagreement in the estimation of the second transition probability between the one in our recent paper (2009 *J. Phys. A: Math. Theor.* **42** 145001) and the other in the comment to it from Baek *et al* (2009 *J. Phys. A: Math. Theor.* **42** 478001). We point out some reasons that the finite size scaling analysis used by them is not proper for the enhanced tree due to its nonamenable nature, which is verified by some numerical results.

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We have recently reported a numerical study of the two-stage bond percolation transition on the enhanced binary tree (EBT) [1]. Two percolation thresholds, $p_{c1} \approx 0.304$ and $p_{c2} \approx 0.56$, which respectively correspond to the divergence of the correlation mass and the correlation length, are obtained. The value of p_{c2} estimated from the fractal exponent $\psi(p)$ is consistent with the duality relation [2], $p_{c2} = 1 - \overline{p}_{c1}$, where $\overline{p}_{c1} \approx 0.436$ is the lower threshold probability of the dual lattice of the EBT. On the other hand, Baek, Minhagen and Kim estimated $p_{c2} \approx 0.48$ for the same model based on the finite size scaling (FSS) analysis [3]. This value is significantly smaller than our estimation while their estimation of p_{c1} and $\overline{p}_{c1}$ is consistent with ours. Thus, they concluded that the duality relation does not hold for the EBT but inequality $p_{c2} < 1 - \overline{p}_{c1}$ is true. Here, we compare these two estimations. In the following, we use p_b to note $p_{c2} \approx 0.48$ obtained in [3] for the distinction.

First, we introduce the scenario of the second transition in the EBT, which has been already shown in [1]. We only assume that the connectedness function, $C_0(\ell, p)$, which is the probability that a site at the ℓth generation belongs to the same cluster with the root site, i.e., the site at zeroth generation, decays as a single exponential function,

$$C_0(\ell, p) = A(p)2^{-\ell/\xi(p)} = A(p)2^{(\psi(p)-1)\ell}, \tag{1}$$

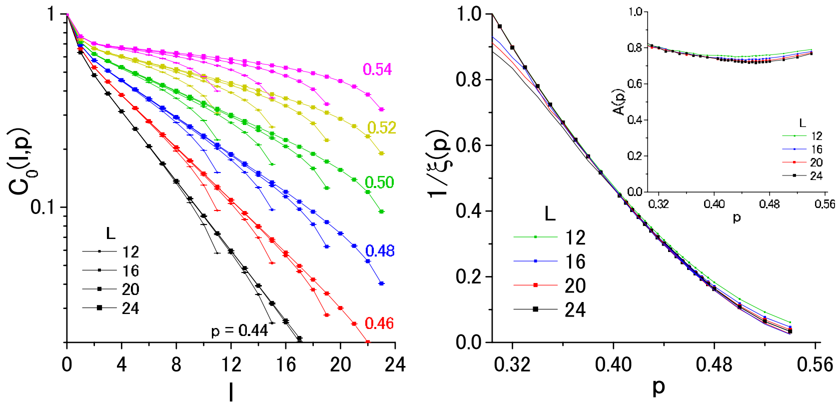

Figure 1. (Left) The connectedness function for six ps and four Ls. Exponential decay can be observed before the boundary effect appears. (Right) p-Dependence of (the inverse of) the correlation length. Symbols indicate the values calculated by $\xi(p, L) = -\log_2[C_0(3L/4, L)/C_0(L/4, p)]/[L/2]$ and dotted lines indicate the values calculated from $1 - \psi(p)$ [1]. The two estimations are almost the same but the former is better near p_{c1} to reproduce $\xi(p_{c1}) = 1$. ξ does not show any singularity around $p = 0.48$ but approaches to zero at $p \approx 0.56$. Inset shows the amplitude, $A(p, L) = C_0(L/2, p)/2^{-L/2\xi(p,L)}$, which hardly depends on p.

for open bond probability $p_{c1} < p < p_{c2}$. Here $\xi(p)$ is a correlation length and $\psi(p) \equiv 1 - 1/\xi(p)$ is a fractal exponent of the divergent clusters. We confirm the exponential decay of $C_0(\ell, p)$ in figure 1. Here we remark on two quantities to detect the second transition,

$$s_0(p, L) \equiv \sum_{\ell=0}^{L-1} 2^\ell C_0(\ell, p) \qquad \text{and} \qquad b(p, L) \equiv 2^{L-1} C_0(L-1, p), \tag{2}$$

where L is a number of generations of finite size samples. We approximately identify $x^L - 1$ with x^L for $x > 1$ in the following, e.g. total number of nodes, $N = 2^L - 1 \to 2^L$. Substitution of equation (1) into equation (2) yields

$$s_0(p, L) = \frac{A(p)}{2^{\psi(p)} - 1} N^{\psi(p)} \qquad \text{and} \qquad b(p, L) = \frac{A(p)}{2^{\psi(p)}} N^{\psi(p)}. \tag{3}$$

In these expressions, $b(p, L)$ and $s_0(p, L)$ have basically the same quantities except unimportant coefficients and then we only treat $b(p, L)$ in the following. Equation (3) leads to an important consequence that b is always infinite in the large size limit, $N \to \infty$, for $p > p_{c1}$.[3] Divergence of $\xi(p)$ at p_{c2}, which is indicated in the right panel of figure 1, results that $\psi(p)$ continuously approaches to unity to produce an $O(N)$ term[4]. What happens at p_{c2} is essentially different from the ordinary second-order transitions in amenable graphs.

Next, we examine the analysis of Baek *et al* [3]. They assumed a FSS formula

$$b(p, L) \propto N^\phi \tilde{f}_3((p - p_b) N^{1/\bar{\nu}}). \tag{4}$$

This formula implies, in a sense of a standard FSS, that b is finite below p_b and diverges as $(p_b - p)^{-\phi\bar{\nu}}$ with infinite N. This seems strange because b has already diverged above $p_{c1}(<p_b)$. Another diverging finite component which results a subleading term in b seems impossible since finite clusters growing with p must be absorbed to the already divergent

[3] The first threshold is defined by $\xi(p_{c1}) = 1$ and then $\psi(p_{c1}) = 0$.

[4] Prefactor $\ell^{-\eta}$ on C_0 is possible but only results a correction factor $(\log N)^{-\eta}$ to s_0 and b.

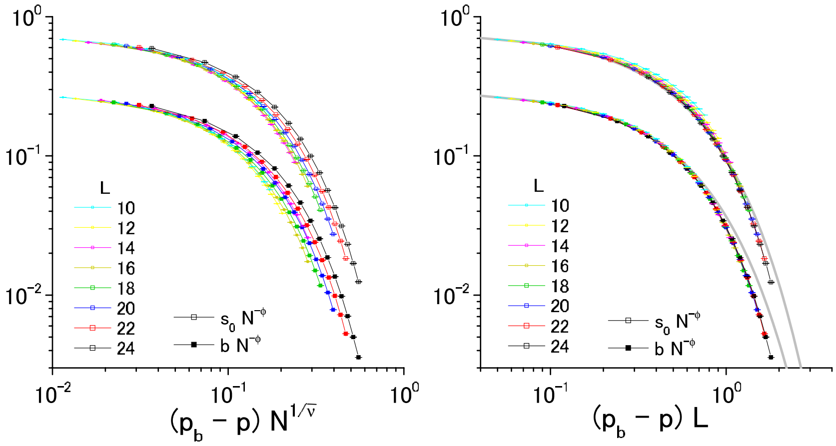

Figure 2. (Left) Finite size scaling (FSS) corresponding to equation (4) using the parameters shown in [3]; $p_b = 0.48$, $\phi = 0.84$ and $1/\overline{\nu} = 0.12$. (Right) FSS corresponding to equation (5) using $p_b = 0.48$, $\phi = 0.84$. We show guidelines proportional to $2^{-3.0(p-p_b)L}$ with light gray color. In both scalings, we use the Monte Carlo data for $0.405 < p < 0.475$ (0.005 step) averaged with 160 000 samples. We show the same FSS of s_b together.

clusters before diverges by themselves. We consider that the scaling behavior is an artifact because equation (4) is approximately reproduced from equation (3) without assuming another diverging component. Equation (3) leads to $b(p, L)/N^{\phi} \propto 2^{(\psi(p_b)-\phi)L+\psi'(p_b)(p-p_b)L+\cdots}$. If one chooses p_b and ϕ satisfying $\phi = \psi(p_b)$, $b(p, L)/N^{\phi}$ looks a function of $(p - p_b)L$ for $|p - p_b| \ll 1$ as

$$b(p, L) \propto N^{\phi} \tilde{g}_3((p - p_b)L). \tag{5}$$

This is obtained by replacing $N^{-1/\overline{\nu}}$ with $L = \log_2 N$ in equation (4). Note that L is locally approximated by a power function $N^{1/\overline{\nu}_{\text{local}}(L)}$ with $\overline{\nu}_{\text{local}}(L) = d \ln L/d \ln N = L \ln 2$, to reproduce equation (4) in a narrow range of L. The two scalings are compared in figure 2. While the scaling with L shows good collapsing of data, the scaling with $N^{1/\overline{\nu}}$ breaks down for large L (we use $1/\overline{\nu} = 0.12$ in [3] and treat larger generations by 7 than that in [3]) and only works in the narrow size range, $L \approx 12$, as predicted from $1/\overline{\nu}_{\text{local}}(12) \approx 0.120$. Note that the scaling with L works for any $p_b \in (p_{c1}, p_{c2})$ if ϕ equals $\psi(p_b)$ (numerically confirmed too, not shown here) and therefore it does not give the threshold of the second transition. Presumably some irrelevant finite size effect or short-range behavior of C_0 yields the best FSS fitting point p_b which depends on the data range of L.

Another evidence for $p_b \approx 0.48$ shown in [3] is the crossing of the ratio of the second largest cluster to the largest cluster, $\langle s_2/s_1\rangle$. Why crossing point gives critical point is based on the fact that the ratio $\langle s_2/s_1\rangle$ in the large size limit behaves as a step function of p around the critical point and takes a special value in the middle of the step on the critical point, which is clearly confirmed by the FSS in the square lattice in [4]. Again it is not clear whether this is also true for the transition of the EBT. If the critical *point* between the non-percolating and percolating phases is replaced by the critical *phase*, characterized by fractional $\psi(p)$, it is naturally expected that a slope appears to fill the gap. Such a slope is actually observed in the Cayley tree for $p_{c1} < p < p_{c2} = 1$ in [4]. Indeed we observe a tendency in the large L limit that $\langle s_2/s_1\rangle$ converges to a value which continuously decreases for $p_{c1} < p < p_{c2}$ rather than forms a step at p_b (not shown here). In addition, we confirmed that $\langle s_2/s_1\rangle$ is far from a universal function of $(p - p_b)N^{1/\overline{\nu}}$ (not shown here) unlike for the case of the square lattice [4]. The crossing of $\langle s_2/s_1 \rangle$ is considered to be caused by the change of the tendency in irrelevant finite size effect.

In conclusion, we provided a simple scenario of the second percolation transition on the EBT and some numerical evidences which support the scenario. We also showed that the FSS performed by Baek *et al* does not hold for a wide range of system sizes. Let us emphasize that the transitions of nonamenable graphs including the EBT are quite different from the usual second-order transitions and standard analysis of second-order transitions in amenable graphs cannot be applied directly to them. The value of p_{c2} is, at least, larger than their estimation and the duality relation, $p_{c2} = 1 - \overline{p}_{c2}$, seems valid for the percolation on the EBT. Baek *et al* also claimed that the duality relation breaks down between the pair of $\{3,7\}$ and $\{7,3\}$ hyperbolic lattices based on the FSS analysis [3]. We consider that they underestimate the second threshold probability in this model too. The duality relation should be true in this model since both of the dual hyperbolic lattices are transitive in the large size limit [2].

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